MATH 6021 Topics in Geometry I - Lecture 1

References:

- * Colding-Minicozzi "A Course in Minimal Surfaces"
- Leon Simon "Lectures on Geometric Measure Theory"
- other literature ......

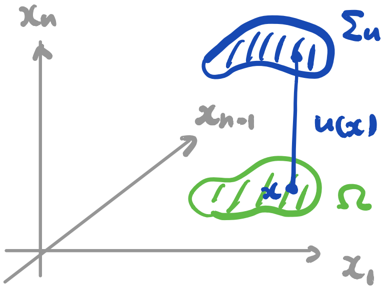
Minimal Surface Theory

- $\Sigma^k \subseteq (M^n, g)$ minimal submanifold (i.e. $\vec{H} \equiv 0$)
- minimizers & unstable critical pts to the area functional
- existence & regularity theory
- geometric & topological applications

The Minimal Surface Equation

(Ref: CM Ch. 1)

Consider the graph of a function $u : \Omega \overset{\text{open}}{\subseteq} \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$



$\Sigma_u := \text{graph}_u$
$= \{(x, u(x)) : x \in \Omega\}$

$$|\Sigma_u| := \text{Area}(\Sigma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx$$

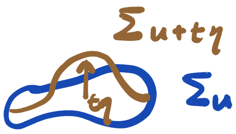
Question: (Plateau)

Given the value of u along $\partial\Omega$, is there a $\text{graph}_u =: \Sigma_u$ with smallest area ?

Note: Such a minimizer (if exists) must be a "critical pt." of $\text{Area}(\Sigma_u)$

1st derivative = 0

1st variation of area (Graphical)



Let $\eta \in C_c^\infty(\Omega)$, compactly supported in Ω.

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_{u+t\eta}| = \left.\frac{d}{dt}\right|_{t=0} \int_\Omega \sqrt{1+|\nabla(u+t\eta)|^2}\,dx$$

$$= \int_\Omega \frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}\,dx \;\overset{\text{Stokes'}}{\underset{\because\, \eta|_{\partial\Omega}=0}{=}}\; -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\cdot\eta\,dx$$

If $\left.\frac{d}{dt}\right|_{t=0} |\Sigma_{u+t\eta}| = 0$ $\forall \eta \in C_c^\infty(\Omega)$, then we have

$$\boxed{\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0}$$

(MSE) in divergence form

E.g. When $n=3$, (MSE) reads

$$\underbrace{(1+u_y^2)\,u_{xx} + (1+u_x^2)\,u_{yy} - 2u_xu_yu_{xy} = 0}_{\text{quasi-linear elliptic 2nd order PDE}}$$

Alternatively, if we define (Recall: $u(x) = u(x_1,\dots,x_{n-1})$)

$$g_{ij} := \delta_{ij} + u_{x_i}u_{x_j} \;\xrightarrow{\text{inverse}}\; g^{ij} = \delta_{ij} - \frac{u_{x_i}u_{x_j}}{1+|\nabla u|^2}$$

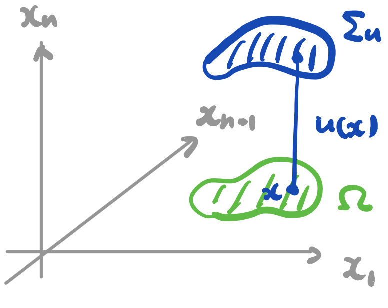
$$\text{(MSE)} \iff \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{u_{x_i}u_{x_j}D_{ij}u}{(1+|\nabla u|^2)^{3/2}} = 0$$

$$\iff \underbrace{\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{1+|\nabla u|^2}\right)}_{g^{ij}} D_{ij}u = 0$$

↑

"$\Delta_{\Sigma_u} x_n = 0$"

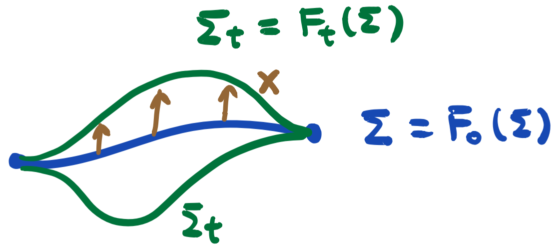
when coord. fcn $x_1,\dots,x_{n-1}$ are harmonic $\Delta_{\Sigma_u} x_i = 0$



Q: non-graphical case ?

First Variation Formula

Let $\Sigma^k \subseteq (M^n, g)$ be a smooth immersed submanifold.



$F : \Sigma \times \overset{t \in}{(-\varepsilon, \varepsilon)} \longrightarrow M$ smooth

s.t.
- $F_t := F(\cdot, t)$ immersion $\forall t$.
- $F_0(\Sigma) = \Sigma$
- $\dot{F}_0 =: X$

$\left(\cdot = \frac{\partial}{\partial t} \right)$

The 1st variation formula is

$$\boxed{\delta \Sigma(X) := \frac{d}{dt}\Big|_{t=0} |\Sigma_t| = \int_\Sigma \operatorname{div}_\Sigma(X)\, dV} \quad \text{——} \quad (*)$$

where $dV :=$ volume measure of Σ

$$\operatorname{div}_\Sigma(X) := \sum_{i=1}^k \langle \nabla_{E_i} X, E_i \rangle$$

Here: $\langle , \rangle =: g \rightsquigarrow$ Levi-Civita connection ∇ on (M, g)

$\{E_i\}_{i=1}^k$ O.N.B. of $T\Sigma$.

Write $X = X^T + X^N \in T\Sigma \oplus N\Sigma$, then $\operatorname{div}_\Sigma(X) = \operatorname{div}_\Sigma X^T + \operatorname{div}_\Sigma X^N$.

Rewrite the "normal part":

$$\operatorname{div}_\Sigma X^N := \sum_{i=1}^k \langle \nabla_{E_i}(X^N), E_i \rangle \overset{\because X^N \perp T\Sigma}{=} - \sum_{i=1}^k \langle X^N, \nabla_{E_i} E_i \rangle$$

$$= - \langle X^N, \sum_{i=1}^k \nabla_{E_i} E_i \rangle = - \langle X, \underbrace{\sum_{i=1}^k (\nabla_{E_i} E_i)^N}_{\vec{H}_\Sigma} \rangle$$

Def$^{\underline{n}}$: $\vec{H}_\Sigma := \sum_{i=1}^k (\nabla_{E_i} E_i)^N$ mean curvature vector of $\Sigma^k \subseteq M^n$

$$(*) \Rightarrow \delta\Sigma(X) = \int_\Sigma \mathrm{div}_\Sigma X^T + \int_\Sigma \mathrm{div}_\Sigma X^N$$

$$= \underbrace{\int_\Sigma \mathrm{div}_\Sigma X^T}_{=0 \text{ if } X|_{\partial\Sigma}=0} - \int_\Sigma \langle X, \vec{H}_\Sigma \rangle$$

Cor: Σ is "stationary" for area among variations fixing $\partial\Sigma$

(or compactly supported if Σ is non-cpt)

$\overset{\text{"def}^{\text{n}}\text{"}}{\Longleftrightarrow}$ $\delta\Sigma(X) = 0$ $\forall$ X vector fields along Σ s.t. $X|_{\partial\Sigma} = 0$

$\Longleftrightarrow$ $\boxed{\vec{H}_\Sigma \equiv 0}$ $\Longleftrightarrow$ Def$^{\text{n}}$: $\Sigma^k \subset (M^n, g)$ minimal submanifold

Remarks: (1) $(*)$ makes sense for "singular" Σ.

(2) Sometimes, $\vec{H}_\Sigma$ is defined with a different sign (or normalized).

(3) $\vec{H}_\Sigma$ is the negative gradient of area functional, hence gives the direction of fastest decrease

$\rightsquigarrow$ Mean Curvature Flow (MCF)

Proof of $(*)$:

Setup: $F : \Sigma \times \overset{t}{\overset{\Uparrow}{(-\varepsilon, \varepsilon)}} \to M$, $F(\cdot, 0) =$ given immersion $\Sigma \hookrightarrow M$.

$X = \dot{F} := \frac{d}{dt}\Big|_{t=0} F(\cdot, t)$ $\qquad$ $F(\Sigma, t) =: \Sigma_t$

$x_1, \dots, x_k$: local coord. on Σ

Write: $g_{ij}(t) := \langle F_{x_i}, F_{x_j} \rangle(t)$ $\quad$ induced metric on Σ_t

$g(t) := (g_{ij}(t))$

$$|\Sigma_t| = \int_\Sigma \sqrt{\det g(t)}\, dx = \int_\Sigma \underbrace{\frac{\sqrt{\det g(t)}}{\sqrt{\det g(0)}}}_{v(t)} \cdot \underbrace{\sqrt{\det g(0)}\, dx}_{dV_\Sigma}$$

Goal: Compute $v'(0)$.

At $t=0$, fix $p \in \Sigma$, assume WLOG, $\underline{g_{ij}(0)(p) = \delta_{ij}}$. Compute at $p \in \Sigma$,

(Recall: $\frac{d}{dt} \log(\det g(t)) = \sum_{i,j=1}^{k} g^{ij}(t)\, \dot{g}_{ij}(t)$)

$$v'(0) = \frac{1}{2} \frac{d}{dt}\Big|_{t=0} \log \det g(t) = \frac{1}{2} \sum_{i,j=1}^{k} \underbrace{g^{ij}(0)}_{\delta^{ij}}\, \dot{g}_{ij}(0) = \frac{1}{2} \sum_{i=1}^{k} \dot{g}_{ii}(0)$$

$$= \sum_{i=1}^{k} \langle \nabla_{F_t} F_{x_i}, F_{x_i} \rangle \overset{\uparrow}{=} \sum_{i=1}^{k} \langle \nabla_{F_{x_i}} F_t, F_{x_i} \rangle = \underbrace{\sum_{i=1}^{k} \langle \nabla_{E_i} X, E_i \rangle}_{\mathrm{div}_\Sigma X\,(p)}$$

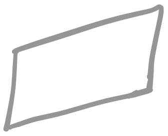
$\because$ $x_1, \dots, x_k, t$ form coord.

$E_i := F_{x_i}(p)$

□

Examples in $\mathbb{R}^3$

(1) Plane

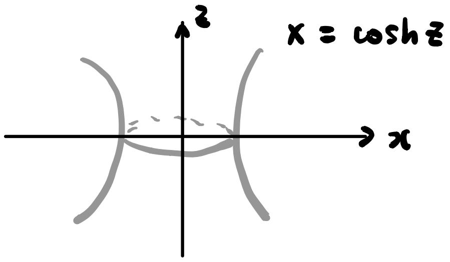


- totally geodesic (i.e. 2^{nd} f.f. $\equiv 0$)
- flat, complete, embedded.
- topologically $\mathbb{R}^2$

(2) Helicoid

- $(t, s) \longmapsto (t\cos s, t \sin s, s)$
- ruled, complete, embedded
- topo. $\approx \mathbb{R}^2$

(3) Catenoid



- rotationally symmetric
- complete, embedded
- topo. $\approx$ annulus

Remark: Up to 1970's, these are the only known examples of complete, minimal embedded surface with finite topology in $\mathbb{R}^3$.

There are many more, c.f. Costa-Hoffman-Meeks, Kapouleas ......

Q: higher dim'l examples?

"complex submanifolds" in $\mathbb{C}^n$

$$\Sigma_{P_1,\dots,P_{n-k}} := \{ (z_1,\dots,z_n) \in \mathbb{C}^n \mid P_1 = \cdots = P_{n-k} = 0 \}$$

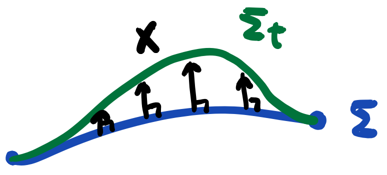
where $P_1,\dots,P_{n-k}$ are complex polynomials.

Recall: Many theorems in Riem. Geom. come from 2^{nd} variation formula for length / energy of geodesics. So, it's natural to look at the 2^{nd} variation for min. submfd as well.

2^{nd} Variation Formula

Let $\Sigma^k \subseteq M^n$ be a min. submfd, i.e. $\vec{H}_\Sigma \equiv 0$.

As in 1^{st} variation, set $\Sigma_t := F_t(\Sigma)$ generated by var. field X



ASSUME: $X \in \Gamma(N\Sigma)$.

i.e. X is normal to Σ

and X compactly supp. away from $\partial\Sigma$

Then, we have:

$$\boxed{\delta^2\Sigma(X) := \frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t| = -\int_\Sigma \langle X, LX \rangle \, dV} \qquad \text{———} \quad (**)$$

where $L : \Gamma(N\Sigma) \longrightarrow \Gamma(N\Sigma)$ is the "Jacobi operator":

$$\mathcal{L}X := \Delta^N_\Sigma X + \sum_{i=1}^k \left(Rm_M(E_i, X)E_i\right)^N + \sum_{i,j=1}^k \langle A_{ij}, X\rangle A_{ij}$$

Here: (i) Δ^N_Σ is the Laplacian on $N\Sigma$, i.e.

$$\Delta^N_\Sigma X := \sum_{i=1}^k (\nabla_{E_i}\nabla_{E_i} X)^N - \sum_{i=1}^k (\nabla_{(\nabla_{E_i}E_i)^T} X)^N$$

(ii) Rm_M = Riem. curvature tensor of (M, g)

(iii) $A_{ij} := (\nabla_{E_i} E_j)^N$ vector-valued 2^{nd} f.f. of Σ

(iv) $\{E_1, \dots, E_k\}$ o.N.B. of $T\Sigma$

Def$^{\underline{n}}$: Σ is stable if $\delta^2\Sigma(X) \geq 0$ $\forall$ cptly supp. X

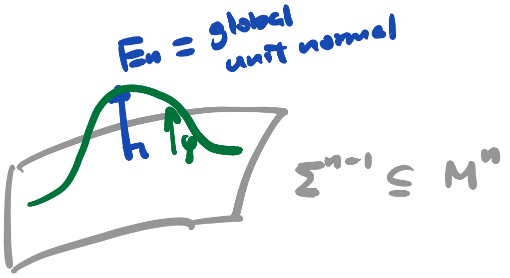
Note: In general, it's difficult to understand $\mathcal{L}$ in higher codimensions (cf. Tsai-Wang 2020)

Hypersurface case ($k = n-1$)

$\exists$ global unit normal E_n on Σ

Assume, further, $N\Sigma$ is trivial (ie $\Sigma^{n-1} \subseteq M^n$ is 2-sided)

$\Rightarrow$ $\mathcal{L}$ becomes a scalar operator (ie acts on functions)



Write: $X = \varphi E_n$ where $\varphi \in C_c^\infty(\Sigma)$

Then,

$$\boxed{\mathcal{L}\varphi = \Delta_\Sigma \varphi + Ric_M(E_n, E_n)\varphi + \|A\|^2 \varphi}$$

$A_{ij} = h_{ij} E_n$

$h_{ij} : T\Sigma \times T\Sigma \to \mathbb{R}$

Remark: (i) $Ric_M > 0 \rightsquigarrow \Sigma$ unstable

(ii) $\|A\|^2 >> 1 \rightsquigarrow \Sigma$ unstable

On the other hand, Σ stable $\Rightarrow$ control on $\|A\|$

(sometimes even pointwise)

Remarks: If Σ is compact, then L has discrete spectrum w.r.t. Dirichlet boundary condition by elliptic PDE theory.

i.e. $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots\cdot \leq \lambda_m \leq \cdots \to +\infty$

Def$^{\underline{n}}$: The **Morse index** of a min. submfd Σ, denoted

$$\text{ind}(\Sigma) := \#\text{ of negative eigenvalues of } L \text{ (w.r.t. Dirichlet condition)}$$

$$= \#\{\lambda_i < 0\}.$$

Note: Σ stable $\Leftrightarrow$ $\text{ind}(\Sigma) = 0$ $\Leftrightarrow$ $\lambda_1(L) \geq 0$